ON POLYNOMIALS BOUNDED AT AN INFINITY OF POINTS

BY

TORD HALL

UPPSALA 1950
APPELBERGS BOKTRYCKERI AB

ON POLYNOMIALS BOUNDED AT AN INFINITY OF POINTS

Inaugural Dissertation

by

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Upsala in April 1950.

Tord Hall

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INTRODUCTION

It is our object in this paper to study polynomials that at given points are bounded by given numbers. We introduce the following notations:

$A_1, A_2, \ldots$ are given positive numbers.

$\lambda_1, \lambda_2, \ldots$ are given points in the complex plane, $\lambda_\nu \neq \lambda_\mu$ when $\nu \neq \mu$.

The point set $\{\lambda_\nu\}$ we denote by E, the derived set of E by E' and the closed set $E+E'$ by $\bar{E}$.

We define by C the class of polynomials $P(z)$ of arbitrary degree, satisfying the inequalities

$$|P(\lambda_\nu)| \leq A_\nu, \quad \nu = 1, 2, \ldots. \tag{1}$$

Finally, for z_0 not belonging to E, we set

$$M(z_0) = \sup_{P \in C} |P(z_0)|.$$

Our aim is to study the quantity $M(z)$. The various possible cases which may occur at a given point z, are classified as follows:

I. The hyperbolic case: $M(z) = 0$.

II. The parabolic case: $0 < M(z) < \infty$.

III. The elliptic case: $M(z) = \infty$.

Naturally, the study of $M(z)$ will mainly be a matter of finding criteria for the cases I, II and III. These criteria will then be in the form of different conditions to be satisfied by the sequences $\{A_\nu\}$ and $\{\lambda_\nu\}$. In the present paper we shall

not concern ourselves with the hyperbolic case, which may be excluded by assuming

$$A_\nu \geq 1, \quad \nu = 1, 2, \ldots. \tag{2}$$

We also suppose

$$\lim_{\nu=\infty} \frac{|\lambda_\nu|^k}{A_\nu} = 0, \quad k = 0, 1, 2, \ldots. \tag{3}$$

The sequences $\{A_\nu\}$ and $\{\lambda_\nu\}$ will satisfy these conditions, except in the final chapter where we shall consider other cases.

We shall now briefly describe our methods and results. First we simplify the problem and deal with the case when all polynomials are of fixed degree and are bounded at a finite number of points. The general case will then be obtained by two limiting processes: in the first we fix the degree n of the polynomials and impose the condition (1) for $\nu = 1, 2, \ldots, m$, $m > n$, and then let $m \to \infty$; in the second we allow the degree of the polynomial to be arbitrary. Using the theory of linear equations [1], we find that the parabolic case (at $z = 0$) will occur if and only if the system of equations

$$\sum_{\nu=1}^{\infty} \frac{\xi_\nu}{A_\nu} \lambda_\nu^k = 0, \quad k = 1, 2, \ldots,$$

has a solution $\{\xi_\nu\}$ such that

$$\sum_{\nu=1}^{\infty} |\xi_\nu| \leq 1,$$

$$\sum_{\nu=1}^{\infty} \frac{\xi_\nu}{A_\nu} \neq 0.$$

Using this criterion, we shall study our original problem by means of Borel series

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[1] F. Riesz: Les systèmes d'équations linéaires à une infinité d'inconnues. Coll. Borel. Paris 1913. Ch. 3.

$$\sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} \cdot \frac{1}{z - \lambda_\nu},$$

where the sequence $\{x_\nu\}$ satisfies the same relations as $\{\xi_\nu\}$.

We so obtain some criteria for the elliptic case.

We shall finally make use of Lagrange's interpolation formula to find a sufficient condition for the parabolic case.

§ 1

GENERAL PROPERTIES OF $M(z)$

1. $M(z)$ from a topological point of view

Already at this point we mention an important property of $M(z)$, which we shall later prove as Theorem VIII. If we denote the complementary set of $\overline{E}$ by ΣD_k, where each D_k is an open connected set, then the following is true for each region D_k:

either $M(z) \equiv +\infty$ *or* $\log M(z)$ *is continuous and subharmonic.*

As an immediate consequence of this theorem we may observe the case when one of the regions D_k, which we call D_∞, contains the point $z = \infty$. By aid of the polynomial $\left(\frac{z}{R}\right)^n$, which belongs to C if $|z| = R$ is a circle, enclosing $\overline{E}$, we obtain

$$M(z) \equiv +\infty \text{ in } D_\infty.$$

2. On the dependence of $M(z)$ on $\{A_\nu\}$ in the parabolic case

We introduce the notations

$$M_A(z_0) = \sup |P(z_0)|, \text{ when } |P(\lambda_\nu)| \leq A_\nu, \quad \nu = 1, 2, \ldots.$$
$$M_B(z_0) = \sup |P(z_0)|, \text{ when } |P(\lambda_\nu)| \leq B_\nu, \quad \nu = 1, 2, \ldots.$$
$$M_{A+B}(z_0) = \sup |P(z_0)|, \text{ when } |P(\lambda_\nu)| \leq A_\nu + B_\nu, \quad \nu = 1, 2, \ldots.$$
$$M_{AB}(z_0) = \sup |P(z_0)|, \text{ when } |P(\lambda_\nu)| \leq A_\nu B_\nu, \quad \nu = 1, 2, \ldots.$$

We call the corresponding classes of polynomials C_A, C_B, C_{A+B}, C_{AB}.

Assuming that $P_1(z) \in C_A$ with $|P_1(z_0)| \geq M_A(z_0) - \varepsilon$, that $P_2(z) \in C_B$ with $|P_2(z_0)| \geq M_B(z_0) - \varepsilon$, and that $P_1(z)$ and $P_2(z)$ have the same argument at the point z_0, we obtain

$$|P_1(\lambda_\nu) + P_2(\lambda_\nu)| \leq |P_1(\lambda_\nu)| + |P_2(\lambda_\nu)| \leq A_\nu + B_\nu, \quad \nu = 1, 2, \ldots,$$

i.e.

$$|P_1(z_0) + P_2(z_0)| \leq M_{A+B}(z_0).$$

On the other hand,

$$M_A(z_0) + M_B(z_0) - 2\varepsilon \leq |P_1(z_0)| + |P_2(z_0)| = |P_1(z_0) + P_2(z_0)|,$$

whence

$$M_A(z_0) + M_B(z_0) \leq M_{A+B}(z_0).$$

Furthermore, we have

$$|P_1(\lambda_\nu)\, P_2(\lambda_\nu)| \leq A_\nu B_\nu, \quad \nu = 1, 2, \ldots,$$

i.e.

$$|P_1(z_0)\, P_2(z_0)| < M_{AB}(z_0).$$

This combined with

$$M_A(z_0)\, M_B(z_0) - \varepsilon' \leq |P_1(z_0)\, P_2(z_0)|$$

gives

$$M_A(z_0)\, M_B(z_0) \leq M_{AB}(z_0).$$

By repeated application of this formula we get, with obvious notation,

$$M^k_{\sqrt[k]{A}}(z_0) \leq \sqrt[k]{M_A(z_0)}.$$

§ 2

POLYNOMIALS OF FIXED DEGREE n, BOUNDED AT MORE THAN n POINTS

3. An introductory theorem

We modify the class of polynomials defined under (1) as follows:

$C_{n,m}$ denotes the class of polynomials $P(z)$ of degree n, satisfying the conditions

$$|P(\lambda_\nu)| \leq A_\nu, \quad \nu = 1, 2, \ldots, m, \quad m > n, \tag{4}$$

where $\lambda_\nu \neq \lambda_\mu$ when $\nu \neq \mu$, and where $\lambda_\nu \neq 0$, $\nu = 1, 2, \ldots, m$.

We then introduce two formal simplifications which do not affect the validity of the results. First we assume that $z_0 = 0$ and put

$$\sup_{P \in C_{n,m}} |P(0)| = M_{n,m}(0) = M_{n,m}. \tag{5}$$

Secondly we suppose that the polynomial $P^*_{n,m}(z) \in C_{n,m}$, where $|P^*_{n,m}(0)| = M_{n,m}$, is real and positive at the origin, i.e.

$$P^*_{n,m}(0) = M_{n,m}. \tag{6}$$

That such a polynomial exists is a consequence of conditions (4), because it follows from (4) that the coefficients of the polynomials in $C_{n,m}$ are uniformly bounded. Therefore, we can find a sequence of polynomials $P_\nu(z)$ in the family such that

(7) $$\lim_{\nu=\infty} P_\nu(z) = P^*_{n,m}(z),$$

Furthermore, let Γ_n be the class of polynomials $P(z)$ of degree n with $P(0) = 1$. We then introduce the positive quantity

(8) $$\mu_{n,m} = \min_{P \in \Gamma_n} \max_{\nu=1,2,\ldots,m} \frac{|P(\lambda_\nu)|}{A_\nu}.$$

Obviously we can here use minimum instead of greatest lower bound, i.e. there exists a polynomial $Q_{n,m}(z)$ solving the problem (8).

We shall now prove the equality

(9) $$\mu_{n,m} M_{n,m} = 1.$$

First we point out that the polynomial $\frac{1}{M_{n,m}} P^*_{n,m}(z)$ belongs to the class Γ_n. Consequently the numbers $\nu = 1, 2, \ldots, m$ surely include a number $\nu = k$ such that

$$\frac{|P^*_{n,m}(\lambda_k)|}{M_{n,m} A_k} \geq \mu_{n,m}.$$

By (4) we get

$$\frac{A_k}{|P^*_{n,m}(\lambda_k)|} \geq 1,$$

hence

(10) $$1 \geq \mu_{n,m} M_{n,m}.$$

For $Q_{n,m}(z)$ on the other hand, we have the inequalities

(11) $$\frac{|Q_{n,m}(\lambda_\nu)|}{\mu_{n,m}} \leq A_\nu, \quad \nu = 1, 2, \ldots, m,$$

i.e. the polynomial $\frac{1}{\mu_{n,m}} Q_{n,m}(z)$ belongs to the class $C_{n,m}$. We thus get the inequality

$$1 \leq \mu_{n,m} M_{n,m},$$

which together with (10) proves (9).

Consequently, if $P^*_{n,m}(z)$ is a polynomial, solving the problem (5), then $\mu_{n,m} P^*_{n,m}(z)$ is a polynomial that solves the problem (8).

We shall now study expressions of the type

$$\sum_{\nu=1}^{m} \frac{|P(\lambda_\nu)|^q}{A_\nu^q} = H_{m,q}(P),$$

where $q > 1$. Since we later shall let $q \to \infty$, we introduce the notation

$$\mu_{n,m}(q) = \min_{P \in \Gamma_n} [H_{m,q}(P)]^{\frac{1}{q}}, \tag{12}$$

which is justified because we subsequently intend to prove

$$\lim_{q=\infty} \mu_{n,m}(q) = \mu_{n,m}.$$

We realize that a polynomial $R_{n,m}(z)$ exists which solves the problem (12). $R_{n,m}(z)$ satisfies the condition

$$\mu_{n,m}(q) = [H_{m,q}(R)]^{\frac{1}{q}}. \tag{13}$$

We next determine the properties of this extremal polynomial by studying the quantity

$$H_{m,q}(R + tU),$$

where $U(z)$ is an arbitrary polynomial of degree $\leq n$ which is zero at the origin, and $t = x + iy$ is a complex variable. Then $H_{m,q}(R + tU)$ will be a real differentiable function of t with a minimum for $t = 0$. Now, developing $H_{m,q}(R + tU)$ in the neighbourhood of the point $t = 0$, we obtain

$$H_{m,q}(R + tU) = H_{m,q}(R) + t\bar{A} + \bar{t}A + o(t),$$

where

$$A = \frac{1}{2}\left(\frac{\partial H}{\partial x} - i\frac{\partial H}{\partial y}\right).$$

In the present case, the minimum of $H_{m,q}(R+tU)$ obviously will be determined by the condition

$$A = 0.$$

As is seen by performing the calculation, $R_{n,m}(z)$ satisfies the relation

$$\sum_{\nu=1}^{m} \frac{|R_{n,m}(\lambda_\nu)|^q}{A_\nu^q} \cdot \frac{U(\lambda_\nu)}{R_{n,m}(\lambda_\nu)} = 0. \tag{14}$$

If $R_{n,m}(\lambda_\nu) = 0$ for some value of ν, the corresponding term in the series (14) will be defined as zero. — For the sake of simplicity we now introduce the notation

$$\alpha_\nu = \left| \frac{R_{n,m}(\lambda_\nu)}{\mu_{n,m}(q)\, A_\nu} \right|^{q-1} \cdot \frac{|R_{n,m}(\lambda_\nu)|}{R_{n,m}(\lambda_\nu)}, \quad \nu = 1, 2, \ldots m, \tag{15}$$

where in conformity with the above we define $\alpha_\nu = 0$ if $R_{n,m}(\lambda_\nu) = 0$. Then, substituting for $U(z)$, first $U(z) = R_{n,m}(z) - 1$ and then $U(z) = z^k$, $k = 1, 2, \ldots n$, (14) may, with the aid of (13) and (15), be written in the following more enlightening form

$$\sum_{\nu=1}^{m} \frac{\alpha_\nu}{A_\nu} = \mu_{n,m}(q), \tag{16}$$

$$\sum_{\nu=1}^{m} \frac{\alpha_\nu}{A_\nu} \lambda_\nu^k = 0, \quad k = 1, 2, \ldots, n, \tag{17}$$

$$\sum_{\nu=1}^{m} |\alpha_\nu|^{\frac{q}{q-1}} = 1. \tag{18}$$

After these introductory steps, we are now able to formulate

Theorem I. *Suppose that*

$$\mu_{n,m}(q) = \operatorname*{Min}_{P \in \Gamma_n} \left\{ \sum_{\nu=1}^{m} \frac{|P(\lambda_\nu)|^q}{A_\nu^q} \right\}^{\frac{1}{q}},$$

and that

$$\varrho_{n,m}(q) = \sup \left| \sum_{\nu=1}^{m} \frac{\xi_\nu}{A_\nu} \right|, \tag{19}$$

where $\xi_1, \xi_2, \ldots \xi_m$ *are numbers satisfying the conditions*

$$\sum_{\nu=1}^{m} \frac{\xi_\nu}{A_\nu} \lambda_\nu^k = 0, \quad k = 1, 2, \ldots n, \tag{20}$$

and

$$\sum_{\nu=1}^{m} |\xi_\nu|^{\frac{q}{q-1}} \leq 1. \tag{21}$$

Then

$$\varrho_{n,m}(q) = \mu_{n,m}(q).$$

Remark. The sequence $\{\alpha_\nu\}$, satisfying the relations (16), (17) and (18), is an extremal sequence that solves the problem (19).

Proof. The numbers α_ν satisfy (20) and (21). Furthermore we have

$$\mu_{n,m}(q) = \sum_{\nu=1}^{m} \frac{\alpha_\nu}{A_\nu}.$$

Then surely

$$\varrho_{n,m}(q) \geq \mu_{n,m}(q). \tag{22}$$

We then assume that $y_1, y_2, \ldots y_m$ are numbers that satisfy (20) and (21) and where, moreover,

$$\varrho_{n,m}(q) = \sum_{\nu=1}^{m} \frac{y_\nu}{A_\nu}.$$

$R_{n,m}(0) = 1$ together with (20) implies that we may write this relation in the form

$$\varrho_{n,m}(q) = \sum_{\nu=1}^{m} \frac{y_\nu}{A_\nu} R_{n,m}(\lambda_\nu).$$

Hölder's inequality then yields

$$\varrho_{n,m}(q) \leq \left\{ \sum_{\nu=1}^{m} |y_\nu|^{\frac{q}{q-1}} \right\}^{\frac{q-1}{q}} \cdot \left\{ \sum_{\nu=1}^{m} \frac{|R_{n,m}(\lambda_\nu)|^q}{A_\nu^q} \right\}^{\frac{1}{q}}.$$

According to (13) and (21) this gives at once

$$\varrho_{n,m}(q) \leq \mu_{n,m}(q),$$

an inequality which, combined with (22), proves the proposition. Evidently, $\{a_\nu\}$ is an extremal sequence solving the problem (19).

4. The limit case $q = \infty$

For different values of q we naturally get different values of $a_\nu = a_\nu(q)$, but since, according to (18),

$$|a_\nu(q)| \leq 1, \quad \nu = 1, 2, \ldots m,$$

there exists a sequence $\{q_i\}$ such that the limits

$$\lim_{i=\infty} a_\nu(q_i) = \beta_\nu, \quad \nu = 1, 2, \ldots m,$$

exist.

We shall now prove the previously mentioned relation

$$\lim_{q=\infty} \mu_{n,m}(q) = \mu_{n,m}.$$

Using (11) and (12) we obtain for the polymomial $Q_{n,m}(z)$

$$\mu_{n,m}(q) \leq \left\{ \sum_{\nu=1}^{m} \frac{|Q_{n,m}(\lambda_\nu)|^q}{A_\nu^q} \right\}^{\frac{1}{q}} \leq \mu_{n,m}\, m^{\frac{1}{q}}. \tag{23}$$

On the other hand, according to (13), we get for $R_{n,m}(z)$

$$\mu_{n,m}(q) = \left\{ \sum_{\nu=1}^{m} \frac{|R_{n,m}(\lambda_\nu)|^q}{A_\nu^q} \right\}^{\frac{1}{q}} \geq \frac{|R_{n,m}(\lambda_\nu)|}{A_\nu}, \quad \nu = 1, 2, \ldots m. \tag{24}$$

From the definition of $\mu_{n,m}$ in (8) it furthermore follows that we can find an integer $\nu = l$ among the numbers $\nu = 1, 2, \ldots m$ such that

$$\frac{|R_{n,m}(\lambda_l)|}{A_l} \geq \mu_{n,m}.$$

In conjunction with (24) this yields

$$\mu_{n,m}(q) \geq \mu_{n,m},$$

which combined with (23) gives the inequalities

$$\mu_{n,m} \leq \mu_{n,m}(q) \leq \mu_{n,m} m^{\frac{1}{q}}.$$

When $q \to \infty$ the result is

$$\lim_{q=\infty} \mu_{n,m}(q) = \mu_{n,m}.$$

Thus, when $q \to \infty$, the relations (16), (17) and (18) become

$$\sum_{\nu=1}^{m} \frac{\beta_\nu}{A_\nu} = \mu_{n,m}, \tag{25}$$

$$\sum_{\nu=1}^{m} \frac{\beta_\nu}{A_\nu} \lambda_\nu^k = 0, \quad k = 1, 2, \ldots n, \tag{26}$$

$$\sum_{\nu=1}^{\infty} |\beta_\nu| = 1. \tag{27}$$

We now observe that

$$\lim_{q=\infty} \varrho_{n,m}(q) = \varrho_{n,m}$$

exists, and from Theorem I we obtain immediately, letting $q \to \infty$:

Theorem II. *Suppose that*

$$\mu_{n,m} = \operatorname*{Min}_{P \in \Pi_n} \operatorname*{Max}_{\nu=1,2,\ldots m} \frac{|P(\lambda_\nu)|}{A_\nu},$$

and that

$$\varrho_{n,m} = \sup \left| \sum_{\nu=1}^{m} \frac{\xi_\nu}{A_\nu} \right|, \tag{28}$$

where $\xi_1, \xi_2, \ldots \xi_m$ *are numbers satisfying the conditions*

$$\sum_{\nu=1}^{m} \frac{\xi_\nu}{A_\nu} \lambda_\nu^k = 0, \quad k = 1, 2, \ldots n, \tag{29}$$

and

$$\sum_{\nu=1}^{m} |\xi_\nu| \leq 1. \tag{30}$$

Then

$$\varrho_{n,m} = \mu_{n,m}.$$

Remark. The sequence $\{\beta_\nu\}$, which satisfies the relations (25), (26) and (27), is an extremal sequence, solving the problem (28).

5. Properties of the extremal sequences and polynomials

Reverting for a moment to finite values of q, we observe that, because of (24), $R_{n,m}(\lambda_\nu) = R_{n,m}(\lambda_\nu; q)$, though depending on q, satisfies

$$|R_{n,m}(\lambda_\nu; q)| \leq A_\nu \mu_{n,m}(q) < K, \quad \nu = 1, 2, \ldots, m,$$

where K is a positive constant. For an appropriate sequence $\{q_i\}$ we thus obtain

$$\lim_{i=\infty} R_{n,m}(\lambda_\nu; q_i) = Q_{n,m}(\lambda_\nu), \quad \nu = 1, 2, \ldots m,$$

where $Q_{n,m}(z)$ is a polynomial solving the problem (8) and at the same time the problem (12) for $q = \infty$.

Considering (15), when $q \to \infty$, we now realize: firstly that $\beta_\nu Q_{n,m}(\lambda_\nu) \geq 0$, $\nu = 1, 2, \ldots m$, and secondly that, if $|Q_{n,m}(\lambda_\nu)| < \mu_{n,m} A_\nu$ for some value of ν, then $\beta_\nu = 0$ for this particular ν-value. — A result concerning a polynomial $Q_{n,m}(z)$ being valid for a polynomial $\mu_{n,m} P^*_{n,m}(z)$, we may formulate our latest results as follows:

$$(31)\quad \begin{cases} \text{(a)}\ \beta_\nu P^*_{n,m}(\lambda_\nu) \geq 0, \ \nu = 1, 2, \ldots m. \\ \text{(b)}\ \text{If}\ |P^*_{n,m}(\lambda_\nu)| < A_\nu, \ \text{then}\ \beta_\nu = 0 \ \text{for this value of}\ \nu. \\ \text{(c)}\ \text{Whenever}\ \beta_\nu \neq 0 \ \text{we have}\ |P^*_{n,m}(\lambda_\nu)| = A_\nu. \end{cases}$$

The equality (31 c) *must hold for at least* $n+1$ *values of* ν: if $|P^*_{n,m}(\lambda_\nu)| = A_\nu$ for only, e.g. $\nu = 1, 2, \ldots n$, then the polynomial $P^*_{n,m}(z) + \varepsilon \prod_{\nu=1}^{n}\left(1 - \frac{z}{\lambda_\nu}\right)$ also would belong to the class $C_{n,m}$. when ε is sufficiently small. At the origin this polynomial would then take the value $M_{n,m} + \varepsilon$, contradicting (5).

6. Uniqueness of the extremal polynomials

We suppose that the ν-values where $\beta_\nu \neq 0$ are $\nu = 1, 2, \ldots p$, where $p \geq n+1$, and that the sequence $\{\beta_\nu\}$ is formed with the aid of the polynomial $P^*_{n,m}(z)$. We moreover suppose that $P(z)$ is a polynomial with the property $P(0) = M_{n,m}$, which also belongs to $C_{n,m}$. Like $P^*_{n,m}(z)$, $P(z)$ satisfies the relations

$$1 = \varrho_{n,m} M_{n,m} = \sum_{\nu=1}^{p} \frac{\beta_\nu}{A_\nu} P(\lambda_\nu) \leq \sum_{\nu=1}^{p} \frac{|P(\lambda_\nu)|}{A_\nu} |\beta_\nu| \leq \sum_{\nu=1}^{p} |\beta_\nu| = 1,$$

which are directly obtained from the propterties of the sequence $\{\beta_\nu\}$ and from the relations (4), (9) and theorem II. Obviously, the sign of equality must hold between all the members. Hence, according to (31 a) and (31 c), $P(z)$ must be equal to $P^*_{n,m}(z)$ at $p \geq n+1$ different points. Consequently, $P(z) \equiv P^*_{n,m}(z)$. From the uniqueness of $P^*_{n,m}(z)$, we directly get the uniqueness of $Q_{n,m}(z)$ and that of the sequence $\{\beta_\nu\}$.

7. An interpolation formula

We shall here deduce a formula, by which an arbitrary polynomial may be expressed as the quotient of two meromorphic functions. We begin with the function

$$\varphi_{n,m}(z) = \sum_{\nu=1}^{m} \frac{\beta_\nu}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu}. \tag{32}$$

According to (26), $\varphi_{n,m}(z)$ may be written

$$\varphi_{n,m}(z) = \sum_{\nu=1}^{m} \frac{\beta_\nu}{A_\nu} \left(\frac{\lambda_\nu}{z}\right)^k \frac{1}{1 - \frac{\lambda_\nu}{z}}, \quad k = 1, 2, \ldots n+1. \tag{33}$$

We now suppose that $P_{n'}(z)$ is an arbitrary polynomial of degree $n' \leq n$, $P_{n'}(z) = a_0 + a_1 z + \ldots a_{n'} z^{n'}$. By successively putting $k = 1, 2, \ldots n'+1$ in (33) and multiplying both sides by a_0, $a_1 z$, $\ldots$, $a_{n'} z^{n'}$ respectively, we obtain a series of equalities

$$a_p z^p \varphi_{n,m}(z) = \sum_{\nu=1}^{m} \frac{\beta_\nu}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu} \cdot a_p \lambda_\nu^p, \quad p = 0, 1, 2, \ldots n'.$$

Adding all these equalities, we get the formula

$$P_{n'}(z) - \frac{\sum_{\nu=1}^{m} \frac{\beta_\nu P_{n'}(\lambda_\nu)}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu}}{\sum_{\nu=1}^{m} \frac{\beta_\nu}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu}} = \frac{\psi_{n,m}(z; P_{n'})}{\varphi_{n,m}(z)}, \quad n' \leq n < m. \tag{34}$$

8. On the zeros of $P^*_{n,m}(z)$ when the points λ_ν lie on the real axis

We apply the formula (34) to $P_{n'}(z) = P^*_{n,m}(z)$ and get

$$P^*_{n,m}(z)\, \varphi_{n,m}(z) = \sum_{\nu=1}^{m} \frac{\beta_\nu P^*_{n,m}(\lambda_\nu)}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu} = \psi_{n,m}(z; P^*),$$

wherein, according to (31 a),

$$\beta_\nu P^*_{n,m}(\lambda_\nu) \geq 0, \quad \nu = 1, 2, \ldots m.$$

Precise information about the distribution of the zeros of $\psi_{n,m}(z; P^*)$ may now be obtained from the following elementary lemma:

Lemma I. *Suppose that the numbers λ_ν are real, and that the numbers a_ν are positive, $\nu = 1, 2, \ldots m$. Then all the zeros of the function*

$$f(z) = \sum_{\nu=1}^{m} a_\nu \frac{\lambda_\nu}{z - \lambda_\nu}$$

are real and simple, and every interval $(\lambda_\nu; \lambda_{\nu+1})$ contains one zero at most.

Proof. Suppose that $f(z_0) = 0$, $z_0 = x_0 + i y_0$. We then get

$$0 = \sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu}{z_0 - \lambda_\nu} = \bar{z}_0 \sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu}{|z_0 - \lambda_\nu|^2} - \sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu^2}{|z_0 - \lambda_o|^2},$$

i.e.

$$x_0 \sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu}{|z_0 - \lambda_\nu|^2} = \sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu^2}{|z_0 - \lambda_\nu|^2} > 0, \tag{35}$$

and

$$y_0 \sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu}{|z_0 - \lambda_\nu|^2} = 0.$$

This gives

$$y_0 = 0.$$

Suppose further that $z_0 = x_0$ is a double zero of $f(z)$. In this case we obtain

$$\sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu}{(x_0 - \lambda_\nu)^2} = 0,$$

contradicting the relation (35).

Suppose, finally, that $x_0 \neq x_1$ are two zeros situated in the same interval $(\lambda_\nu; \lambda_{\nu+1})$. From

$$\sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu}{x_0 - \lambda_\nu} = 0 \quad \text{and} \quad \sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu}{x_1 - \lambda_\nu} = 0$$

we get

$$\sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu}{(x_0 - \lambda_\nu)(x_1 - \lambda_\nu)} = 0.$$

Subtracting this equality from

$$\sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu}{(x_0 - \lambda_\nu)} \cdot \frac{1}{x_1} = 0,$$

we get

$$\sum_{\nu=1}^{m} \frac{a_\nu \lambda_\nu^2}{(x_0 - \lambda_\nu)(x_1 - \lambda_\nu)} = 0,$$

which is absurd.

Remark. More exact information about the zeros may be obtained from a graph of the two functions in the right member of the equality

$$\frac{1}{x} f(x) = \sum_{\nu=1}^{m} \frac{a_\nu}{x - \lambda_\nu} - \frac{1}{x} \sum_{\nu=1}^{m} a_\nu.$$

The lemma applied to our case now gives

Theorem III. *If the points λ_ν lie on the real axis, then:*

1. *All the zeros of $P^*_{n,m}(z)$ are real and simple.*
2. *Every interval $(\lambda_\nu; \lambda_{\nu+1})$ contains at most one zero of $P^*_{n,m}(z)$.*

Remark. An analogous result is naturally valid for $\varphi_{n,m}(z)$.

§ 3

POLYNOMIALS OF ARBITRARY DEGREE. BOUNDED AT AN INFINITY OF POINTS

9. The limit case $m = \infty$

We first state one more elementary lemma, which is required in the following:

Lemma II. *Suppose that* $\xi_{\nu,m}$ *are numbers satisfying the conditions*

1. $$\sum_{\nu=1}^{\infty} |\xi_{\nu,m}| \leq 1, \quad m = 1, 2, \ldots,$$

2. $$\lim_{m=\infty} \xi_{\nu,m} = \xi_\nu, \quad \nu = 1, 2, \ldots,$$

and suppose that a_ν *are given numbers restricted by the condition*

$$\lim_{\nu=\infty} a_\nu = 0.$$

Then

$$\lim_{m=\infty} \sum_{\nu=1}^{\infty} a_\nu \xi_{\nu,m} = \sum_{\nu=1}^{\infty} a_\nu \xi_\nu.$$

Next we point out four obvious inequalities concerning $M_{n,m}$ and $\varrho_{n,m}$, namely

$$\begin{cases} M_{n,m} \geq M_{n,m+1}, \\ \varrho_{n,m} \leq \varrho_{n,m+1}, \text{ when } n \text{ is fixed,} \end{cases}$$

and

$$\begin{cases} M_{n,m} \leq M_{n+1,m}, \\ \varrho_{n,m} \geq \varrho_{n+1,m}, \text{ when } m \text{ is fixed.} \end{cases}$$

By (2) we have $A_\nu \geq 1$. This gives $M_{n,m} \geq 1$ and $\varrho_{n,m} \leq 1$. Consequently $\lim_{m=\infty} M_{n,m} \geq 1$ and $\lim_{m=\infty} \varrho_{n,m} > 0$ both exist.

We now hold n fixed and let $m \to \infty$.

Theorem IV. *Suppose that*

$$M_n = \sup_{P \in C_n} |P(0)|, \tag{36}$$

where C_n is the class of polynomials $P(z)$ of degree n, satisfying the conditions

$$|P(\lambda_\nu)| \leq A_\nu, \quad \nu = 1, 2, \ldots,$$

and suppose that

$$\varrho_n = \sup \left| \sum_{\nu=1}^{\infty} \frac{\xi_\nu}{A_\nu} \right|, \tag{37}$$

where the numbers ξ_ν satisfy the conditions

$$\sum_{\nu=1}^{\infty} \frac{\xi_\nu}{A_\nu} \lambda_\nu^k = 0, \quad k = 1, 2, \ldots n, \tag{38}$$

and

$$\sum_{\nu=1}^{\infty} |\xi_\nu| \leq 1. \tag{39}$$

Then

$$\varrho_n = \frac{1}{M_n}.$$

Remark. There always exists an extremal sequence $\{\gamma_\nu\}$, which satisfies the relations

$$\sum_{\nu=1}^{\infty} \frac{\gamma_\nu}{A_\nu} = \varrho_n, \tag{40}$$

$$\sum_{\nu=1}^{\infty} \frac{\gamma_\nu}{A_\nu} \lambda_\nu^k = 0, \quad k = 1, 2, \ldots n, \tag{41}$$

$$\sum_{\nu=1}^{\infty} |\gamma_\nu| = 1. \tag{42}$$

Proof. We first state that

$$\lim_{m=\infty} M_{n,m} = M_n. \tag{43}$$

This is clear from the fact that, according to (3), the numbers A_ν increase towards infinity more rapidly than the values at λ_ν of any fixed polynomial. Consequently, as soon as m exceeds a certain value we have $M_{n,m} = M_n$.

The next step is to prove that

$$\lim_{m=\infty} \varrho_{n,m} = \varrho_n. \tag{44}$$

We start with the extremal sequence $\{\beta_\nu\}$ that solves the problem (28). We select a sequence (m_i) such that the limits

$$\lim_{i=\infty} \beta_\nu(m_i) = \gamma_\nu, \quad \nu = 1, 2, \ldots,$$

exist. According to (3) we may apply lemma II, which gives

$$\lim_{m=\infty} \varrho_{n,m} = \lim_{i=\infty} \varrho_{n,m_i} = \lim_{i=\infty} \sum_{\nu=1}^{m_i} \frac{\beta_\nu(m_i)}{A_\nu} = \sum_{\nu=1}^{\infty} \frac{\gamma_\nu}{A_\nu},$$

and

$$\lim_{i=\infty} \sum_{\nu=1}^{m_i} \frac{\beta_\nu(m_i)}{A_\nu} \lambda_\nu^k = \sum_{\nu=1}^{\infty} \frac{\gamma_\nu}{A_\nu} \lambda_\nu^k = 0, \quad k = 1, 2, \ldots n.$$

Further, we get

$$\sum_{\nu=1}^{\infty} |\gamma_\nu| \leq 1.$$

We have thus proved the existence of a sequence $\{\gamma_\nu\}$, satisfying (38) and (39), where

$$\sum_{\nu=1}^{\infty} \frac{\gamma_\nu}{A_\nu} = \lim_{m=\infty} \varrho_{n,m}.$$

By definition of ϱ_n in (37) we then get

$$\varrho_n \geq \lim_{m=\infty} \varrho_{n,m}. \tag{45}$$

On the other hand, from an infinite sequence $\{\xi_\nu\}_1^\infty$, satisfying (38) and (39), we may construct a finite sequence $\{\xi'_\nu = \xi_\nu + \delta_\nu\}_1^m$, satisfying (29) and (30), and such that, moreover,

$$\sum_{\nu=1}^{m} \frac{\xi'_\nu}{A_\nu} = \varrho_n - \varepsilon_m.$$

This is always possible because the quantities $\left| \sum_{\nu=m+1}^{\infty} \frac{\xi_\nu}{A_\nu} \right|$ and $\left| \sum_{\nu=m+1}^{\infty} \frac{\xi_\nu}{A_\nu} \lambda_\nu^k \right|$, $k = 1, 2, \ldots n$, must of course become arbitrarily small as m increases to infinity. By definition of $\varrho_{n,m}$ in (28) we then get

$$\varrho_n - \varepsilon_m \leq \varrho_{n,m},$$

i.e.

$$\varrho_n \leq \lim_{m=\infty} \varrho_{n,m}.$$

Together with (45) this proves (44).

From (9) and theorem II we know that

$$\varrho_{n,m} = \frac{1}{M_{n,m}}.$$

This, combined with (43) and (44), proves the proposition

$$\varrho_n = \frac{1}{M_n}.$$

Finally we may point out that the extremal sequence $\{\gamma_\nu\}$ must satisfy the relation

$$\sum_{\nu=1}^{\infty} |\gamma_\nu| = 1.$$

10. Further results in the limit case $m = \infty$

We observe that

$$\lim_{m=\infty} P^*_{n,m}(z) = P^*_n(z),$$

with

$$P^*_n(0) = M_n$$

exists, and that $P^*_n(z)$, as well as the sequence $\{\gamma_\nu\}$, is uniquely determined. As in (31) we have

(46)
$$\begin{cases} \text{(a)} & \gamma_\nu P^*_n(\lambda_\nu) \geq 0, \quad \nu = 1, 2, \ldots \\ \text{(b)} & \text{If } |P^*_n(\lambda_\nu)| < A_\nu, \text{ then } \gamma_\nu = 0 \text{ for this } \nu\text{-value.} \\ \text{(c)} & \text{Whenever } \gamma_\nu \neq 0 \text{ we have } |P^*_n(\lambda_\nu)| = A_\nu, \text{ and this holds for at least } n+1 \text{ values of } \nu. \end{cases}$$

Actually, when n is fixed, we always deal with finite sequences $\{\gamma_\nu\}$. — From (32) we obtain for $m \to \infty$

$$\lim_{m=\infty} \varphi_{n,m}(z) = \varphi_n(z) = \sum_{\nu=1}^{\infty} \frac{\gamma_\nu}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu}.$$

Analogously, (34) becomes

(47)
$$P_{n'}(z) = \frac{\sum_{\nu=1}^{\infty} \frac{\gamma_\nu P_{n'}(\lambda_\nu)}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu}}{\sum_{\nu=1}^{\infty} \frac{\gamma_\nu}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu}} = \frac{\psi_n(z; P_{n'})}{\varphi_n(z)}.$$

for $n' \leq n$.

Finally, we see that lemma I holds for $m \to \infty$ if the series with positive terms

$$\sum_{\nu=1}^{\infty} a_\nu < \infty.$$

Consequently, we may generalize theorem III to get

Theorem V. *If the points λ_ν lie on the real axis, then:*

1. *All the zeros, $z_1, z_2, \ldots z_n$, of $P_n^*(z)$ are real and simple.*
2. *If, furthermore,*

$$\sum_{\nu=1}^{\infty} \frac{1}{|\lambda_\nu|} < \infty,$$

then

$$\sum_{\nu=1}^{n} \frac{1}{|z_\nu|} \text{ is uniformly bounded as } n \to \infty.$$

Remark. An analogous result is naturally valid for $\varphi_n(z)$.

11. Polynomials of arbitrary degree

We shall here first generalize theorem IV and then prove theorem VIII, which we have already mentioned at the beginning of this paper.

Theorem VI. *Suppose that*

$$M = \sup_{P \in C} |P(0)|,$$

C being the class of polynomials $P(z)$ of arbitrary degree, that satisfies the inequalities

$$|P(\lambda_\nu)| \leq A_\nu, \quad \nu = 1, 2, \ldots,$$

where λ_ν are given points in the complex plane, $\lambda_\nu \neq \lambda_\mu$ when $\nu \neq \mu$,
and where $A_\nu \geq 1$ are numbers such that

$$\lim_{\nu=\infty} \frac{|\lambda_\nu|^k}{A_\nu} = 0, \quad k = 0, 1, 2, \ldots$$

Suppose, furthermore, that

$$\varrho = \sup \left| \sum_{\nu=1}^{\infty} \frac{\xi_\nu}{A_\nu} \right|, \tag{48}$$

where the numbers ξ_ν *satisfy the conditions*

$$\sum_{\nu=1}^{\infty} \frac{\xi_\nu}{A_\nu} \lambda_\nu^k = 0, \quad k = 1, 2, \ldots, \tag{49}$$

and

$$\sum_{\nu=1}^{\infty} |\xi_\nu| \leq 1. \tag{50}$$

Then

$$\varrho = \frac{1}{M}.$$

Remark. If $\varrho > 0$, there exists an extremal sequence $\{x_\nu\}$ such that

$$\sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} = \varrho, \tag{51}$$

$$\sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} \lambda_\nu^k = 0, \quad k = 1, 2, \ldots, \tag{52}$$

$$\sum_{\nu=1}^{\infty} |x_\nu| = 1. \tag{53}$$

Proof. We first state the inequalities $M_n \leq M_{n+1}$ and $\varrho_n \geq \varrho_{n+1}$. Consequently the sequence $\{\varrho_n\}$ is bounded below, which implies that $\lim_{n=\infty} \varrho_n \geq 0$ exists. — By definition of M we further have

$$\lim_{n=\infty} M_n = M.$$

We then proceed as in theorem IV. By aid of the extremal sequence $\gamma_\nu = \gamma_\nu(n)$, $\nu = 1, 2, \ldots$, we establish the existence of

$$x_\nu = \lim_{i=\infty} \gamma_\nu(n_i), \quad \nu = 1, 2, \ldots,$$

where the numbers x_ν satisfy the conditions (49) and (50) and where

$$\sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} = \lim_{n=\infty} \varrho_n .$$

The definition of ϱ in (48) then gives

$$\varrho \geq \lim_{n=\infty} \varrho_n .$$

For every n, however, we evidently have $\varrho \leq \varrho_n$. Hence

$$\varrho \leq \lim_{n=\infty} \varrho_n ,$$

and we obtain

$$\varrho = \lim_{n=\infty} \varrho_n .$$

From theorem IV we know that

$$\varrho_n = \frac{1}{M_n} .$$

Consequently

$$\varrho = \frac{1}{M} .$$

A corollary of theorem VI is

Theorem VII. *The necessary and sufficient condition for the occurence of the parabolic case at the origin is that the system of equations*

$$\sum_{\nu=1}^{\infty} \frac{\xi_\nu}{A_\nu} \lambda_\nu^k = 0 , \quad k = 1 , 2 , \ldots ,$$

has a solution $\{\xi_\nu\}$, $\sum_{\nu=1}^{\infty} |\xi_\nu| \leq 1$, *for which*

$$\sum_{\nu=1}^{\infty} \frac{\xi_\nu}{A_\nu} \neq 0 .$$

For the proof of theorem VIII we require the following facts. Supposing $\varrho > 0$, we obtain for $n \to \infty$

$$\lim_{n=\infty} \varphi_n(z) = \varphi(z) = \sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu}, \tag{54}$$

and

$$\lim_{n=\infty} \psi_n(z; P_{n'}) = \psi(z; P_{n'}) \sum_{\nu=1}^{\infty} \frac{x_\nu P_{n'}(\lambda_\nu)}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu},$$

where $\varphi(z)$ and $\psi(z; P_{n'})$ are obviously regular in the complementary set of $\bar{E}$. If now $P(z)$ is a polynomial, belonging to the class C, the formula (47) becomes

$$P(z) = \frac{\sum_{\nu=1}^{\infty} \frac{x_\nu P(\lambda_\nu)}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu}}{\sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu}}. \tag{55}$$

Theorem VIII. *If* $\sum D_k$ *denotes the complementary set of* $\bar{E}$, *then in each of the regions* D_k, *either:*

1. $M(z) \equiv +\infty$, *or*
2. $\log M(z)$ *is continuous and subharmonic.*

Proof. If $M(z) \not\equiv +\infty$ in D_k, we must have one point — say $z = 0$ — where $M(0) < \infty$. We shall first prove that this implies that $M(z)$ must be bounded in every closed region D'_k, interior to D_k. If this is not true, then, for $P(z) \in C$,

$$\sup_{z \in D'_k} |P(z)| = +\infty.$$

Thus, there exists a point $z_0 \in D'_k$, limit-point of a sequence $\{z_{0i}\}$, with the property

$$\lim_{z_{0i} \to z_0} |P_{n_i}(z_{0i})| = +\infty \tag{56}$$

for certain $P_{n_i}(z) \in C$. Now, let $2\delta > 0$ denote the shortest

distance between D'_k and the boundary of D_k, and let C_ν denote the circle $|z-z_0|=\delta_\nu$, $\delta>\delta_1>\delta_2>\ldots\to 0$. From the maximum-modulus theorem together with (56) we then get

$$\overline{\lim_{i=\infty}} \max_{z\in C_\nu} |P_{n_i}(z)| = +\infty,$$

i.e. there exists on every circle C_ν one point z_ν, limit-point of a sequence $\{z_{\nu i}\}$, such that

$$\overline{\lim_{z_{\nu i}\to z_\nu}} |P_{n_i}(z_{\nu i})| = +\infty, \quad \nu = 1, 2, \ldots. \tag{57}$$

We shall now use the formula (55) for the representation of $P_{n_i}(z)$ in D_k,

$$P_{n_i}(z)\,\varphi(z) = \sum_{\nu=1}^{\infty} \frac{x_\nu P_{n_i}(\lambda_\nu)}{A_\nu} \cdot \frac{\lambda_\nu}{z-\lambda_\nu}.$$

For $|z-z_0|<\delta$ and $|z-\lambda_\nu|>\delta$, $\nu=1, 2, \ldots,$ we obtain $\left|\frac{\lambda_\nu}{z-\lambda_\nu}\right| = \left|1-\frac{z}{z-\lambda_\nu}\right| < 1+\frac{|z_0|+\delta}{\delta} = K$, where K is a positive constant. This, combined with the relations $|P_{n_i}(\lambda_\nu)|\le A_\nu$, $\nu=1, 2, \ldots,$ and $\sum_{\nu=1}^{\infty}|x_\nu| = 1$ gives

$$|\varphi(z)| < \frac{K}{|P_{n_i}(z)|}.$$

Since $\varphi(z)$ is regular in D_k we get from (56) and (57)

$$\varphi(z_\nu) = 0, \quad \nu = 0, 1, 2, \ldots.$$

Consequently, the zeros of $\varphi(z)$ have a limit-point in D_k, which leads to $\varphi(z)\equiv 0$ in D_k. For $z=0$ we then obtain

$$\varphi(0) = -\sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} = -\varrho = 0,$$

i.e. $M(0) = \infty$ contrary to the hypothesis. We thus find that $M(z)$ is bounded in each closed sub-region of D_k, if we have $M(z) \not\equiv +\infty$ in D_k.

In order to prove the continuity of $M(z) \geq 1$ we observe that, for $z_1 \in D'_k$ and $|z - z_1| < \delta$, we have

$$|M(z) - M(z_1)| \leq \sup_{P \in C} |P(z) - P(z_1)| \leq K_\delta |z - z_1|, \; K_\delta < \infty,$$

the first inequality following from the definition of $M(z)$, and the second from the well-known fact that, if the members of a class of analytic functions are uniformly bounded in every closed region D'_k, interior to a given region D_k, then the same property holds for their derivatives.

Finally, we conclude that $\log M(z)$ is subharmonic, since

$$\log M(z) = \sup_{P \in C} \log |P(z)| \leq \sup_{P \in C} \frac{1}{2\pi} \int_0^{2\pi} \log |P(z + \delta e^{i\Theta})| \, d\Theta <$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \log M(z + \delta e^{i\Theta}) \, d\Theta .$$

§ 4

SOME ELLIPTIC CASES

We shall now apply the results obtained to some special cases that occur when the sequences $\{A_\nu\}$ and $\{\lambda_\nu\}$ are restricted by various conditions.

12. The points λ_ν lie on a Jordan curve

Theorem IX. *Suppose that all the points λ_ν lie on a Jordan curve, enclosing the origin; if we only suppose that the numbers A_ν satisfy the condition*

$$\lim_{\nu=\infty} A_\nu = \infty,$$

then we have the elliptic case $M = \infty$.

This fact, $M = M(0) = \infty$, combined with Theorem VIII, yields that $M(z) \equiv \infty$ inside the curve; according to the remark on p. 10, the same holds outside the curve.

For the proof of the theorem we shall use a lemma about Borel series:

Lemma III. *Suppose that*

1. *$z_1, z_2, \ldots$ are points situated on a Jordan curve Γ, which encloses the region D, containing $z = 0$.*

2. *$c_1, c_2, \ldots$ are numbers such that*

$$\sum_{\nu=1}^{\infty} |c_\nu| < \infty. \tag{58}$$

3.

$$\frac{c_0}{z} + \sum_{\nu=1}^{\infty} \frac{c_\nu}{z - z_\nu} \equiv 0 \text{ outside } \Gamma. \tag{59}$$

Then

$$c_\nu = 0, \quad \nu = 0, 1, 2, \ldots .$$

Proof. We shall first establish that, if $w = w(z)$, where $w(0) = 0$, is the function which represents the region D on the unit circle $|w| < 1$, and if $w_\nu = w(z_\nu)$, $\nu = 1, 2, \ldots$, we have

$$\frac{c_0}{w} + \sum_{\nu=1}^{\infty} \frac{c_\nu}{w - w_\nu} \equiv 0 \text{ for } |w| > 1.$$

From (59) we directly get

$$\frac{1}{z}\left\{ c_0 + \sum_{\nu=1}^{\infty} c_\nu \right\} + \frac{1}{z^2} \sum_{\nu=1}^{\infty} c_\nu z_\nu + \ldots \frac{1}{z^k} \sum_{\nu=1}^{\infty} \frac{c_\nu z_\nu^{k-1}}{1 - \frac{z_\nu}{z}} \equiv 0.$$

This gives

$$(60) \qquad \begin{cases} c_0 + \sum_{\nu=1}^{\infty} c_\nu = 0 \\ \sum_{\nu=1}^{\infty} c_\nu z_\nu^k = 0, \quad k = 1, 2, \ldots . \end{cases}$$

Now we shall avail ourselves of a generalization of a classical theorem due to Runge[1]: Suppose that $f(z)$ is analytic in the region D, which is bounded by the Jordan curve Γ, and continuous in the closed region $\bar{D} = D + \Gamma$. Then it is always possible to approximate $f(z)$ uniformly in $\bar{D}$ by a polynomial $P(z)$, i.e.

$$|f(z) - P(z)| < \varepsilon \text{ in } \bar{D}.$$

We put $f(z) = [w(z)]^k$ and obtain

$$(61) \qquad |w_\nu^k - P(z_\nu)| < \varepsilon, \quad \nu = 1, 2, \ldots,$$

where $P(0) = 0$. We then proceed from the identity

$$\sum_{\nu=1}^{\infty} c_\nu w_\nu^k = \sum_{\nu=1}^{\infty} c_\nu P(z_\nu) + \sum_{\nu=1}^{\infty} c_\nu [w_\nu^k - P(z_\nu)].$$

[1] J. L. Walsh: Interpolation and Approximation by Rational Functions in the Complex Domain. Am. Math. Soc. Coll. Publ. Vol. XX. N. Y. 1935. P. 36.

According to (60) and to the fact that $P(0)=0$, we have

$$\sum_{\nu=1}^{\infty} c_\nu P(z_\nu)=0.$$

By means of (61) and (58) we then get

$$\left|\sum_{\nu=1}^{\infty} c_\nu w_\nu^k\right| < \varepsilon \sum_{\nu=1}^{\infty} |c_\nu| < \varepsilon',$$

i.e.

$$\sum_{\nu=1}^{\infty} c_\nu w_\nu^k = 0, \quad k=1, 2, \ldots.$$

This in conjunction with (60) gives

$$\frac{c_0}{w}+\sum_{\nu=1}^{\infty}\frac{c_\nu}{w-w_\nu}\equiv 0 \text{ for } |w|>1. \tag{62}$$

We now must prove that $c_n=0$, where n is an arbitrary positive integer. To that end we consider the term

$$\frac{c_n}{w-w_n}$$

and put $w=re^{i\Theta}$, where $1<r<1+\delta$, $\delta>0$, and $\Theta=\arg w_n$. Around the point w_n we describe a circle C' with the radius δ to be given below, and, furthermore, we divide the series in (62) into three parts

$$\frac{c_0}{w}+\frac{c_n}{w-w_n}+\sum_{0<|w_n-w_\nu|<\delta}\frac{c_\nu}{w-w_\nu}+\sum_{|w_n-w_\nu|\geq\delta}\frac{c_\nu}{w-w_\nu}=0.$$

By elementary geometry we obtain

$$|w-w_n|=r-1,$$

$|w-w_\nu|>r-1$ for all w_ν within the circle C' except w_n,

$|w-w_\nu|>\delta$ for all w_ν outside or on C'.

We thus have

$$\frac{c_n}{r-1} < \frac{1}{r-1} \sum_{0<|w_n-w_\nu|<\delta} |c_\nu| + \frac{1}{\delta} \sum_{|w_n-w_\nu|\geq\delta} |c_\nu| + \frac{|c_0|}{r}.$$

Since

$$\sum_{\nu=1}^{\infty} |c_\nu| < K < \infty,$$

it is possible, for every n, to find a δ such that

$$\sum_{0<|w_n-w_\nu|<\delta} |c_\nu| < \frac{\varepsilon}{2},$$

whence

$$|c_n| < \frac{\varepsilon}{2} + (r-1)\left[\frac{K}{\delta} + \frac{|c_0|}{r}\right].$$

Having thus fixed δ, we may choose r, $1 < r < 1+\delta$, so that

$$r - 1 < \frac{\varepsilon}{2\left[\frac{K}{\delta} + \frac{|c_0|}{r}\right]},$$

i.e.

$$|c_n| < \varepsilon.$$

Consequently

$$c_n = 0, \quad n = 1, 2, \ldots,$$

and we then get $c_0 = 0$ too.

We now return to the proof of theorem IX.

Proof. Suppose on the contrary, that $M < \infty$. In this case we have $\frac{1}{M} = \varrho > 0$ and hence the extremal sequence $\{x_\nu\}$, satisfying the relations (51), (52) and (53), exists. We form the function

$$f(z) = -\frac{\varrho}{z} + \sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} \cdot \frac{1}{z-\lambda_\nu}.^{1}$$

---

[1] We may observe that $f(z) = \frac{1}{z}\varphi(z)$, where $\varphi(z)$ is the function defined by (54).

From (51) and (52) we get

$$f(z) = \sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} \left(\frac{\lambda_\nu}{z}\right)^k \frac{1}{z - \lambda_\nu}, \quad k = 1, 2, \ldots .$$

For $|z| > R > \sup_{\nu=1,2,\ldots} |\lambda_\nu| = \lambda$ we then obtain

$$|f(z)| < K \left[\frac{\lambda}{R}\right]^k, \quad k = 1, 2, \ldots .$$

K being a positive constant. This gives

$$f(z) \equiv 0 \text{ for } |z| > R,$$

and hence $f(z) \equiv 0$ throughout the region outside Γ. Lemma III then yields $x_\nu = 0$, $\nu = 1, 2, \ldots$, and $\varrho = \frac{1}{M} = 0$, which contradicts the assumption $M < \infty$. Consequently we must have $M = \infty$.

13. The set $\{\lambda_\nu\}$ is bounded

We shall now weaken the conditions satisfied by the points λ_ν, and strengthen the conditions satisfied by the numbers A_ν. We can prove

Theorem X. *Suppose that*

1. $A_1 \leq A_2 \leq \ldots < A_\nu \leq \ldots$,

2. $\overline{\lim_{n=\infty}} \sqrt[n]{A_n} > 1$,

3. $|\lambda_\nu| < R$, $\nu = 1, 2 \ldots .$

Then the elliptic case, $M = \infty$, occurs.

This result, $M(z) = \infty$ for every $z \neq \{\lambda_\nu\}$, is especially interesting when z belongs to parts of the plane where the set $\{\lambda_\nu\}$ is dense.

Proof. We argue as in theorem IX: We suppose that $M < \infty$ and form the function

$$f(z) = -\frac{\varrho}{z} + \sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} \cdot \frac{1}{z - \lambda_\nu};$$

we find as before that $f(z) \equiv 0$ for $|z| > R$.

Introducing the notation

$$r_n = \sum_{\nu=n+1}^{\infty} \frac{|x_\nu|}{A_\nu}$$

we then get, according to a theorem due to Beurling [1], that every $x_\nu = 0$, if

$$\overline{\lim_{n=\infty}} \sqrt[n]{r_n} < 1.$$

Here our assumptions yield

$$r_n = \sum_{\nu=n+1}^{\infty} \frac{|x_\nu|}{A_\nu} \leq \frac{1}{A_{n+1}} \sum_{\nu=n+1}^{\infty} |x_\nu| \leq \frac{1}{A_{n+1}} \leq \frac{1}{A_n},$$

hence

$$\overline{\lim_{n=\infty}} \sqrt[n]{r_n} \leq \frac{1}{\overline{\lim_{n=\infty}} \sqrt[n]{A_n}} < 1.$$

Consequently every $x_\nu = 0$, and, as in theorem IX, we get $M = \infty$.

14. The points λ_ν lie in a half-plane

Using a well-known theorem due to Carleman [2], we are able to prove

Theorem XI. *Suppose that all the points λ_ν are situated in the half-plane $I\{z\} \leq -1$. Then the condition*

[1] A. Beurling: Sur les fonctions limites quasi analytiques des fractions rationelles. Skand. Mat. Kongr. Stockholm 1934.

[2] T. Carleman: Les fonctions quasi analytiques. Coll. Borel. Paris 1926. Ch. 3.

$$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{c_k}} = \infty,$$

where

$$c_k = \sup_{\nu=1,2,\ldots} \frac{|\lambda_\nu|^k}{A_\nu}, \quad k = 1, 2, \ldots,$$

implies $M = \infty$.

Proof. As above, we suppose that $M < \infty$ and start with the function

$$f(z) = -\frac{\varrho}{z} + \sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} \cdot \frac{\lambda_\nu}{z - \lambda_\nu},$$

or

$$f(z) = \sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} \left(\frac{\lambda_\nu}{z}\right)^k \frac{1}{z - \lambda_\nu}, \quad k = 1, 2, \ldots$$

We then consider $f(z)$ in the half-plane $I\{z\} > 1$. We obtain

$$|f(z)| < \frac{1}{|z|^k} \sum_{\nu=1}^{\infty} |x_\nu| \frac{|\lambda_\nu|^k}{A_\nu} < \frac{\sup_{\nu=1,2,\ldots} \frac{|\lambda_\nu|^k}{A_\nu} \cdot \sum_{\nu=1}^{\infty} |x_\nu|}{|z|^k},$$

hence

$$|f(z)| < \left| \frac{\sqrt[k]{c_k}}{z} \right|^k, \quad k = 1, 2, \ldots$$

Applying Carleman's theorem, we conclude $f(z) \equiv 0$ for $I\{z\} > 1$. Thus

$$f(z) = -\frac{\varrho}{z} + \sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} \cdot \frac{1}{z - \lambda_\nu} \equiv 0$$

wherever $f(z)$ is holomorphic and so in the neighbourhood of $z = 0$, where $f(z)$ has an isolated pole. This is absurd. Consequently $\varrho = 0$ and $M = \infty$.

As a special case of theorem IX we obtain

Theorem XII. *Suppose that all the points λ_ν lie on the real axis, and that*

$$1. \quad 0 < |\lambda_1| < |\lambda_2| < \ldots < |\lambda_\nu| < \ldots \to \infty,$$

$$2. \qquad A_\nu = \frac{|\lambda_\nu|^\nu}{|\lambda_1 \lambda_2 \ldots \lambda_\nu|}, \quad \nu = 1, 2, \ldots$$

Then

$$\sum_{\nu=1}^{\infty} \frac{1}{|\lambda_\nu|} = \infty$$

assures that $M = \infty$.

Proof. We obtain

$$c_k = |\lambda_1 \lambda_2 \ldots \lambda_k|,$$

and since

$$\frac{1}{|\lambda_k|} < \frac{1}{\sqrt[k]{|\lambda_1 \lambda_2 \ldots \lambda_k|}},$$

the result follows immediately.

15. The polynomials $\prod_{\nu=1}^{n}\left(1-\frac{z}{\lambda_\nu}\right)$

Polynomials with zeros at the points λ_ν and belonging to C are of great importance. This is shown by

Theorem XIII. *Suppose that the class C contains an infinite number of polynomials*

$$P(z) = \prod_{\nu=1}^{n}\left(1-\frac{z}{\lambda_\nu}\right).$$

Then the elliptic case, $M = \infty$, occurs.

Proof. As before we suppose that $M < \infty$. From the properties of the extremal sequence $\{x_\nu\}$ and from the assumption we get

$$\varrho = \sum_{\nu=1}^{\infty} \frac{x_\nu}{A_\nu} P(\lambda_\nu) = \sum_{\nu=n+1}^{\infty} \frac{x_\nu}{A_\nu} P(\lambda_\nu) \leq \sum_{\nu=n+1}^{\infty} |x_\nu| = \varepsilon_n,$$

i.e. $\varrho = 0$ and $M = \infty$.

The two last theorems of this paragraph are consequences of theorem XIII and are at the same time closely connected with theorem XII.

Theorem XIV. *Suppose that the numbers λ_ν are real and positive,*

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_\nu < \ldots \to \infty.$$

and suppose that

$$A_\nu = \frac{\lambda_\nu^\nu}{\lambda_1 \cdot \lambda_2 \ldots \lambda_\nu}, \quad \nu = 1, 2, \ldots.$$

Then $M = \infty$.

Proof. We have to establish that

$$P(z) = \prod_{\nu=1}^{n} \left(1 - \frac{z}{\lambda_\nu}\right) \in C,$$

i.e.

$$|P(\lambda_\nu)| \leq \frac{\lambda_\nu^\nu}{\lambda_1 \cdot \lambda_2 \ldots \lambda_\nu}, \quad \nu = 1, 2, \ldots$$

For $\nu = 1, 2, \ldots n$, we have $P(\lambda_\nu) = 0$. For $\nu = m > n$ we get

$$|P(\lambda_m)| = \prod_{\nu=1}^{n} \left(\frac{\lambda_m}{\lambda_\nu} - 1\right) < \prod_{\nu=1}^{n} \frac{\lambda_m}{\lambda_\nu} = \frac{\lambda_m^n}{\lambda_1 \cdot \lambda_2 \ldots \lambda_n},$$

i.e.

$$|P(\lambda_m)| < \frac{\lambda_m^n}{\lambda_1 \cdot \lambda_2 \ldots \lambda_m} \cdot \frac{\lambda_m^{m-n}}{\lambda_{n+1}\, \lambda_{n+2} \ldots \lambda_m} = \frac{\lambda_m^m}{\lambda_1 \cdot \lambda_2 \ldots \lambda_m}.$$

Hence $P(z) \in C$ and, according to theorem XIII, we have $M = \infty$.

If the numbers λ_ν are real and situated symmetrically in relation to the origin, we may modify theorem XIV as follows:

Theorem XV. *Suppose that*

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_\nu < \ldots \to \infty,$$

and that the class C consists of polynomials $P(z)$ of arbitrary degree, satisfying the inequalities

$$|P(\pm \lambda_\nu)| \leq \left[\frac{\lambda_\nu^\nu}{\lambda_1 \cdot \lambda_2 \ldots \lambda_\nu}\right]^2, \quad \nu = 1, 2, \ldots.$$

Then we have the elliptic case, $M = \infty$.

Proof. As in theorem XIV we find that the polynomials $\prod_{\nu=1}^{n} \left(1 - \frac{z^2}{\lambda_\nu^2}\right)$ belong to C. Theorem XIII then gives $M = \infty$.

§ 5

A PARABOLIC CASE

16. In this paragraph, another method will be used to find a sufficient condition for the parabolic case. We shall not make use of the relations (2) and (3) and we start with *polynomials of fixed degree n, bounded at n + 1 points.*

From (31 c) and the remark following it, we get

$$|P^*_{n,n+1}(\lambda_\nu)| = A_\nu, \quad \nu = 1, 2, \ldots n+1. \tag{63}$$

For a polynomial $P(z)$ of degree n, satisfying (63), Lagrange's interpolation formula gives

$$\left\{\begin{array}{l} P(z) = \omega_{n+1}(z) \displaystyle\sum_{\nu=1}^{n+1} \frac{A_\nu e^{i\Theta_\nu}}{(z-\lambda_\nu)\,\omega'_{n+1}(\lambda_\nu)}, \\ \text{where} \\ \omega_{n+1}(z) = \displaystyle\prod_{k=1}^{n+1}\left(1 - \frac{z}{\lambda_\nu}\right). \end{array}\right. \tag{64}$$

For $z = 0$ we obtain

$$|P(0)| \leq \sum_{\nu=1}^{n+1} \frac{A_\nu}{|\lambda_\nu\,\omega'_{n+1}(\lambda_\nu)|}.$$

The sign of equality holds, if we choose

$$\Theta_\nu = \pi + \arg\left[\lambda_\nu\,\omega'_{n+1}(\lambda_\nu)\right], \quad \nu = 1, 2, \ldots n+1. \tag{65}$$

Evidently, the polynomial $P^*_{n,n+1}(z)$, where $P^*_{n,n+1}(0) = M_{n,n+1}$, must satisfy this condition. We thus get

$$M_{n,n+1} = \sum_{\nu=1}^{n+1} \frac{A_\nu}{|\lambda_\nu \omega'_{n+1}(\lambda_\nu)|} = \sum_{\nu=1}^{n+1} \frac{A_\nu}{\prod_{k=1,2,\ldots n+1}^{k\neq\nu} \left|1 - \frac{\lambda_\nu}{\lambda_k}\right|}$$

We may point out that this formula can also be deduced from the system of equations (25), (26) and (27), if $m = n+1$.

For $M_{n,n+1}$ we have the inequalities $M_{n,n+1} \geq M_{n,n+2} \geq \geq M_{n,n+3} \geq \ldots$. Consequently

$$M_n \leq M_{n,n+1}.$$

If $M_{n,n+1} < K < \infty$, $n = 1, 2, \ldots$, then $\overline{\lim_{n=\infty}} M_{n,n+1} = M'$ is finite and we get $M \leq M'$, i.e. *if* $M' < \infty$, *then surely* $M < \infty$. We now suppose that $\sum_{\nu=1}^{\infty} \frac{1}{|\lambda_\nu|} < \infty$. Then

$$\lim_{n=\infty} \omega_{n+1}(z) = \omega(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{\lambda_\nu}\right)$$

exists and is an integral function. Furthermore we suppose

$$M' = \sum_{\nu=1}^{\infty} \frac{A_\nu}{|\lambda_\nu \omega'(\lambda_\nu)|} < \infty.$$

Our result may then be summed up in

Theorem XVI. *Suppose that the set* $\{\lambda_\nu\}$ *contains an infinity of points such that*

$$\sum_{\nu=1}^{\infty} \frac{1}{|\lambda_\nu|} < \infty,$$

and suppose that the numbers A_ν *satisfy the conditions*

$$\overline{\lim_{\nu=\infty}} A_\nu > 0$$

and

$$\sum_{\nu=1}^{\infty} \frac{A_\nu}{|\lambda_\nu \omega'(\lambda_\nu)|} < \infty,$$

where

$$\omega(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{\lambda_\nu}\right).$$

Then the parabolic case, $0 < M < \infty$, *occurs.*

In order to illustrate the exactness of this theorem for certain distributions of the points λ_ν, we choose $\lambda_\nu = 2^\nu$, $\nu = 1, 2, \ldots$ We obtain

$$|2^\nu \omega'(2^\nu)| = (c + o(1)) \cdot 2^{\frac{\nu(\nu-1)}{2}}, \text{ where } c > 0.$$

If we then choose

$$A_\nu = 2^{\frac{\nu(\nu-1)}{2} - \varepsilon\nu}, \ \varepsilon > 0, \ \nu = 1, 2, \ldots,$$

theorem XVI implies

$$0 < M < \infty,$$

while if we choose

$$A_\nu = 2^{\frac{\nu(\nu-1)}{2}}, \ \nu = 1, 2, \ldots,$$

theorem XV gives $M = \infty$.